Economic predictions with big data: The illusion of sparsity

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Predictive modeling with big data

$$y_t = \beta_1 x_{1t} + \cdots + \beta_k x_{kt} + \varepsilon_t\ ,\ \ \varepsilon_t \underset{iid}{\sim} \mathcal{N}(0, \sigma^2)$$

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- Standard inference (ML or flat prior) is not viable
  - Proliferation of parameters
  - High estimation uncertainty
  - Overfitting and imprecise out-of-sample forecasting / poor external validity

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➥ Methods to address curse of dimensionality increasingly popular
  - **Sparse** modeling e.g. hand picking, Lasso regression
  - **Dense** modeling e.g. Ridge regression, Factor models

This paper: Sparse or dense modeling?

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- Answer is an empirical matter
  - Study a variety of predictive problems in macro, micro and finance

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- Answer is an empirical matter
  - Study a variety of predictive problems in macro, micro and finance
- Popular techniques not suitable to answer the question
  - Sparsity/density often assumed
  - A small set of predictors might be selected simply to reduce estimation error, even if the model is not sparse

This paper: Sparse or dense modeling?

- Answer is an empirical matter
  - Study a variety of predictive problems in macro, micro and finance

- Our predictive model
  - **sparsity**, without assuming it
  - **shrinkage**, to give a chance to large models
  - Bayesian inference on sparsity and shrinkage

Main results

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   - Posterior not concentrated on a single sparse model, but on a wide set

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⬇

The illusion of sparsity

Outline

- The predictive model
- Applications to macro, micro and finance
- Sparse or dense modeling?
  - Exploring the posterior

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- Prior

$$\beta_i | \sigma^2, \gamma^2, q \underset{iid}{\sim} \begin{cases} \mathcal{N}(0, \sigma^2\gamma^2) & \text{with probability} \quad q \\ 0 & \text{with probability} \quad 1-q \end{cases}$$

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Probability of inclusion, controls **size**

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Variance of prior, controls **shrinkage**

Probability of inclusion, controls **size**

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- “Spike-and-slab” prior
  - Mitchel and Beauchamp (1988)
  - Vast literature on Bayesian Model Averaging and Variable Selection

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  - This paper: inference on $\boldsymbol{q}$ and $\boldsymbol{\gamma^2}$

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- Hyperpriors

$\propto Var(y_t | q, \gamma^2, \sigma^2)$

$\propto Var(x_t' \beta | q, \gamma^2, \sigma^2)$

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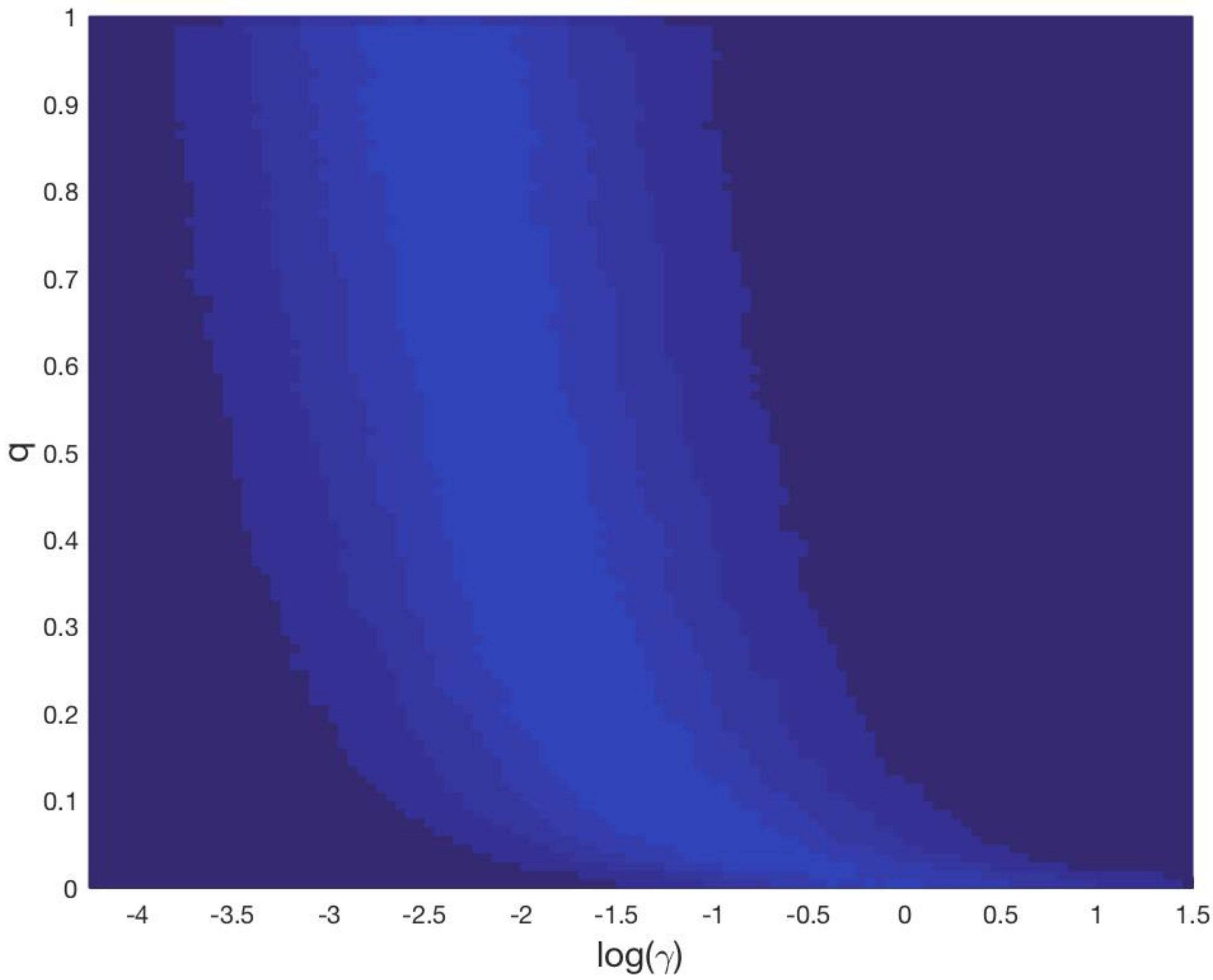
The implied joint prior on q and γ^2



The prior distribution

- Alternative representation

$$\beta_i | \sigma^2, \gamma^2, q \underset{iid}{\sim} \mathcal{N}(0, \sigma^2 \gamma^2 z_i),$$

$$z_i \underset{iid}{\sim} Bernoulli(q)$$

- Relation with other popular shrinkage methods
  - Ridge regression: $q = 1$
  - Lasso regression: $z_i \underset{iid}{\sim} \mathcal{E}xponential$
  - Horse shoe prior: $z_i \underset{iid}{\sim} Half\ Cauchy$

  - None admits a truly sparse representation of with positive probability

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Economic applications

- Macro
  - Forecasting industrial production with many macro predictors
  - The determinants of economic growth in a cross-section of countries
- Finance
  - Prediction of the US equity premium over time
  - Explaining the cross-section of equity returns across firms
- Micro
  - Understanding the decline in crime rates in US states during the 1990s
  - The determinants of government takings of private properties in US judicial circuits
- Some references:
  - Stock-Watson (2002a and b), Barro-Lee (1994), Sala-i-Martin et al. (2004), Welch-Goyal (2008), Freyberger et al. (2017), Donohue-Levitt (2001), Chen-Yeh (2012), Belloni et al. (2011, 2012, 2014).

Economic applications

	Y	X	Sample
Macro 1	Growth rate of US **Industrial Prod.**	**130** lagged macro and financial indicators	**659 time-series** obs. Feb. 60-Dec. 14
Macro 2	**Countries average growth** 1960-1985	**60** country charact' socio-econ, inst.	**90 cross-section** obs.
Finance 1	US **equity premium**	**16** lagged macro and financial indicators	**58 time-series** obs. 1948-2015
Finance 2	**Stock returns** of US firms	**144** dummies lagged characts'	**≈1400k panel** obs. Jul. 63–Dec. 15, ≈2k firms
Micro 1	**Crime rate** in US states	**285** state characts' socio-econ, inst., law	**476 panel** obs. Jan. 86–Dec. 97, 48 states
Micro 2	**Eminent domain** decisions	**138** judges' characts' socio-polit., profess	**312 panel** obs. 1975-2008, circuits

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Exploring the posterior

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   - Posterior not concentrated on a single sparse model, but on a wide set
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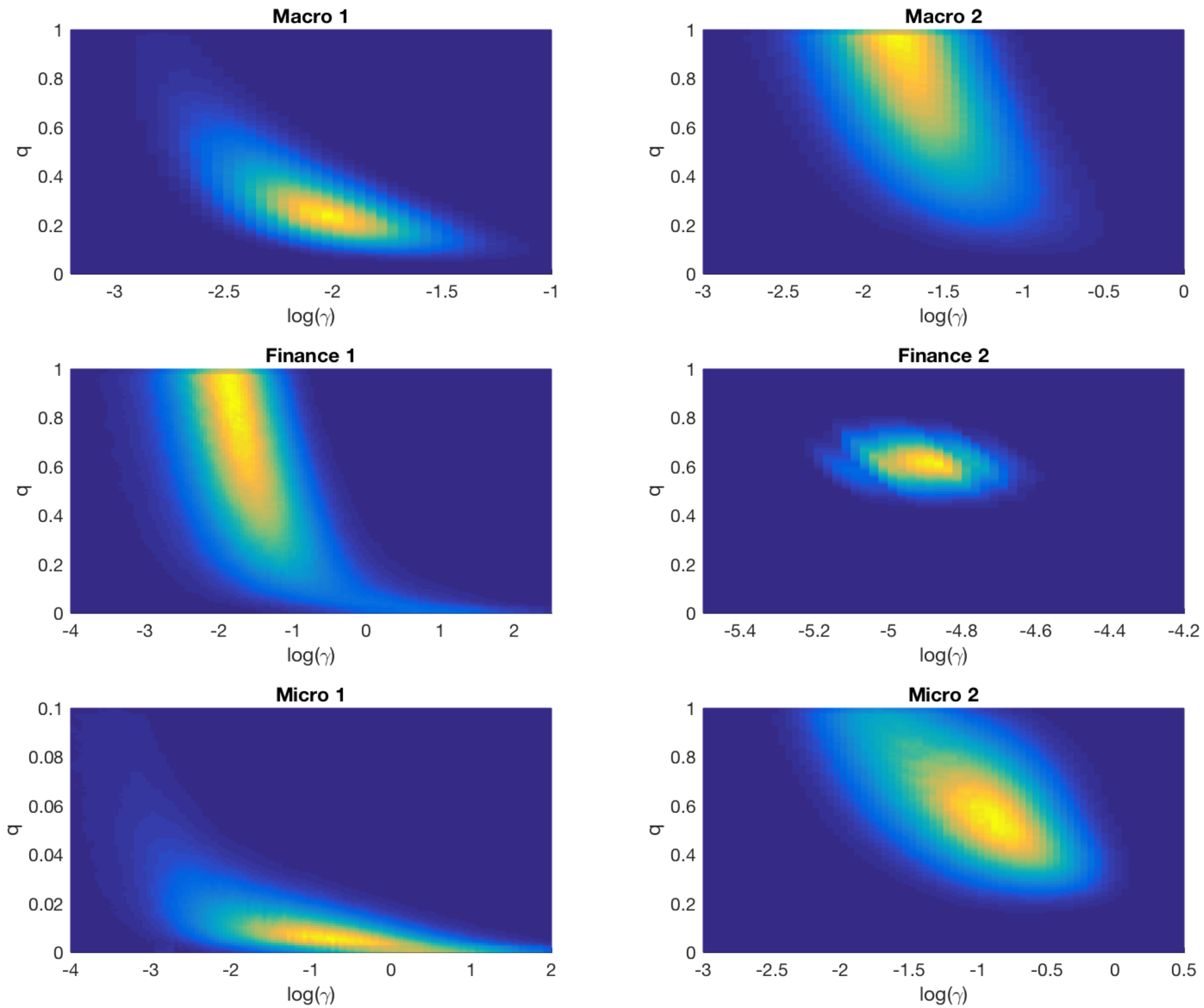
Variance of prior, controls **shrinkage**

Probability of inclusion, controls **size**

Probability of inclusion and prior variance (q and γ^2)



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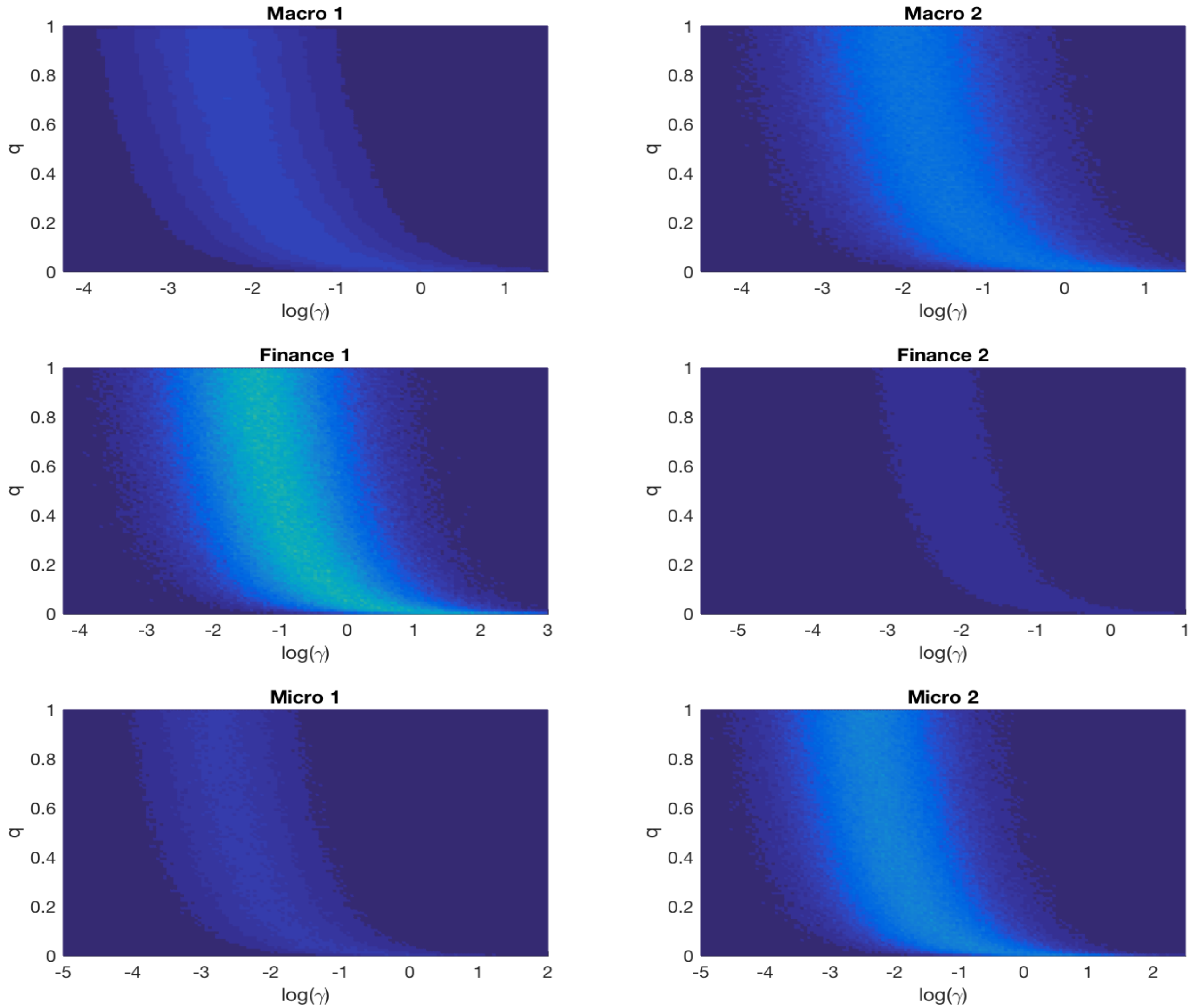

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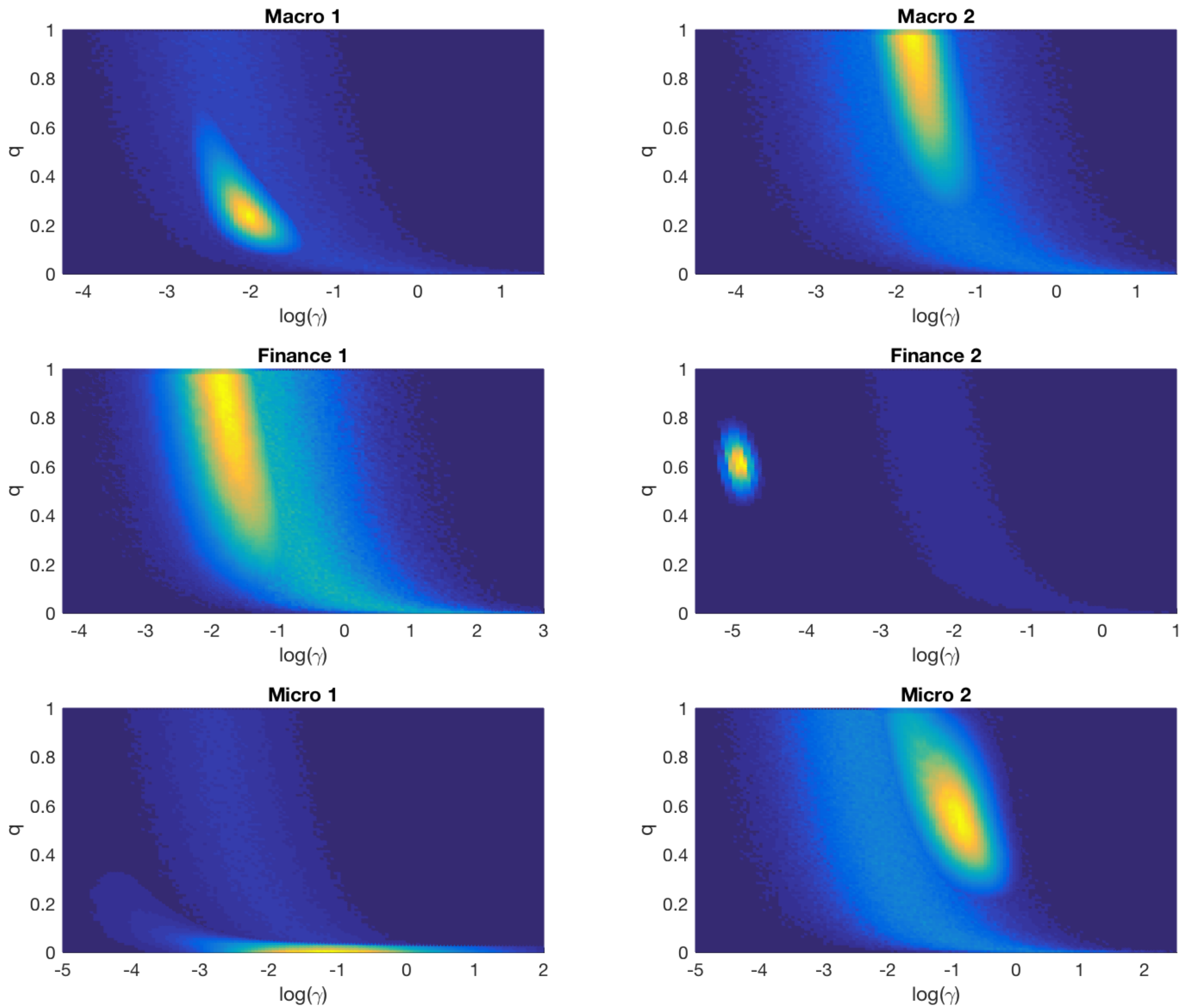
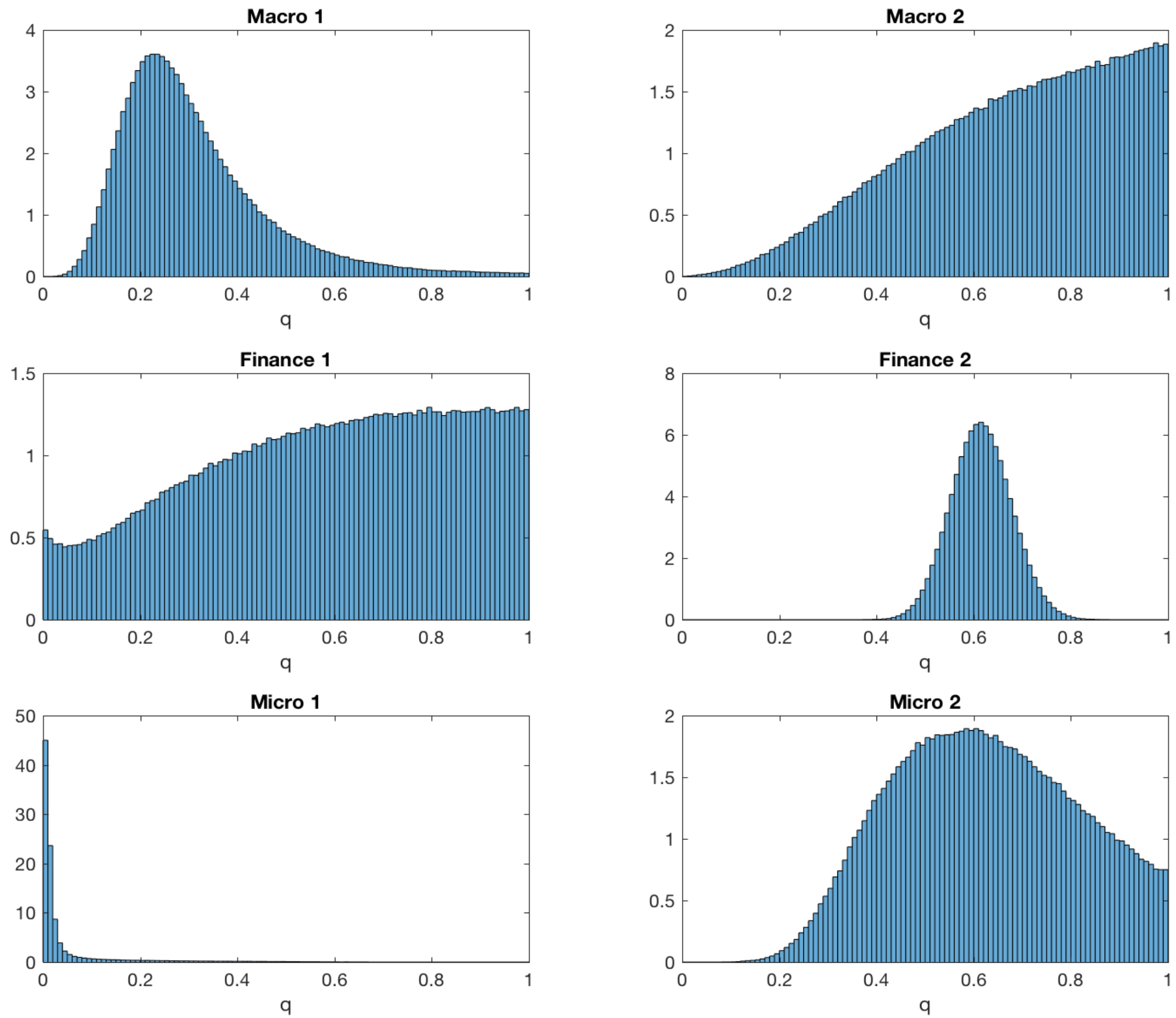


Posterior probability of inclusion: $p(q|Y)$



$p(q|Y)$ as a measure of predictive accuracy

- Posterior of q

$$p(q|Y) \propto p(Y|q) \cdot p(q)$$

$p(q|Y)$ as a measure of predictive accuracy

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$$p(q|Y) \propto p(Y|q)$$

$p(q|Y)$ as a measure of predictive accuracy

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$$p(q|Y) \propto p(Y|q) = \prod_{t}^{T} p(y_t|y^{t-1}, q)$$

$p(q|Y)$ as a measure of predictive accuracy

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$$p(q|Y) \propto p(Y|q) = \prod_{t}^{T} p(y_t|y^{t-1}, q)$$

predictive score

$p(q|Y)$ as a measure of predictive accuracy

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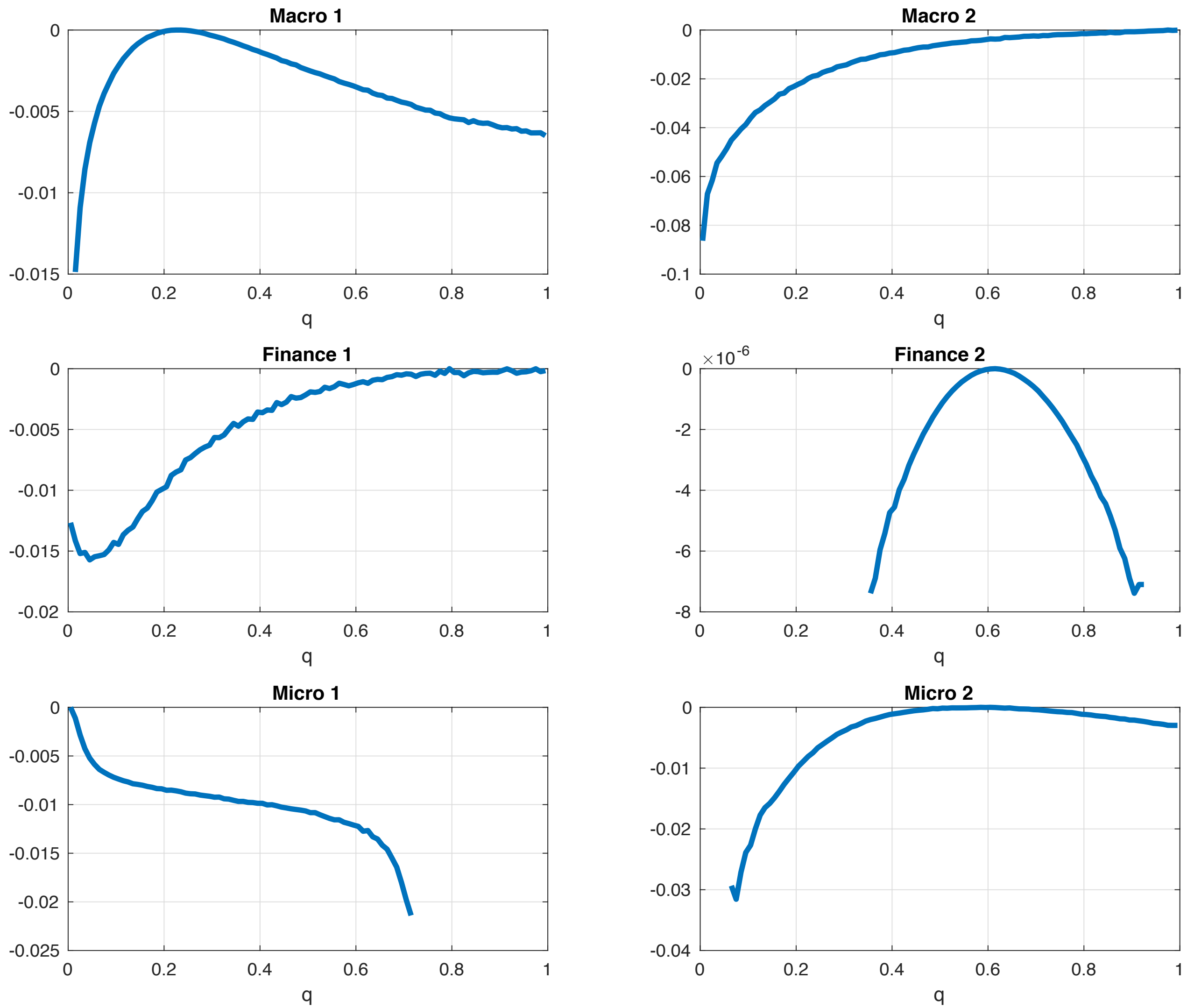
➥ Average log-predictive score

$$\frac{1}{T}\sum_{t}^{T} \log p(y_t|y^{t-1}, q) \ = \frac{1}{T} log\, p(q|Y) \ + \ \ constant$$

Average log-predictive score, relative to best fitting model



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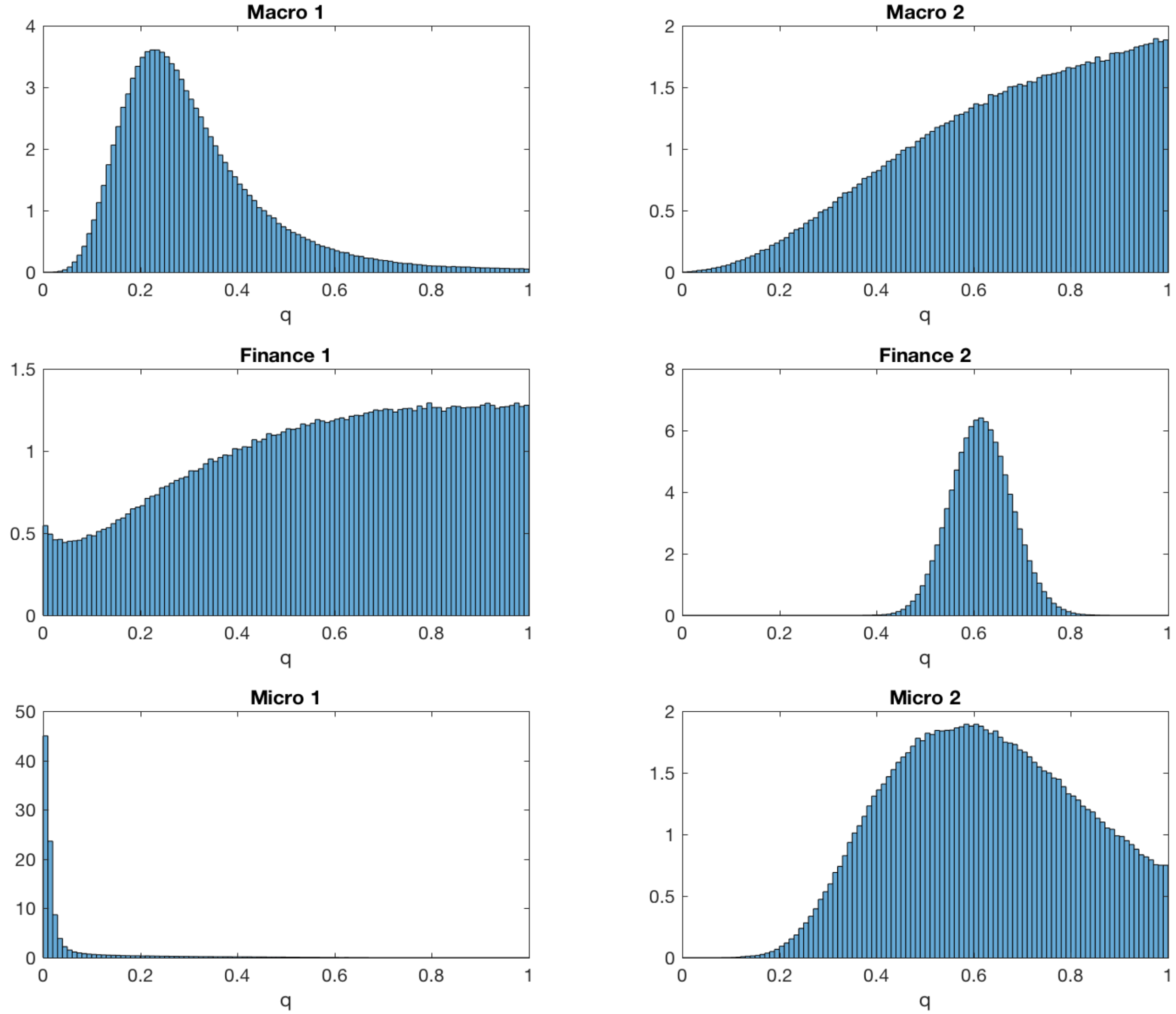

Exploring the posterior

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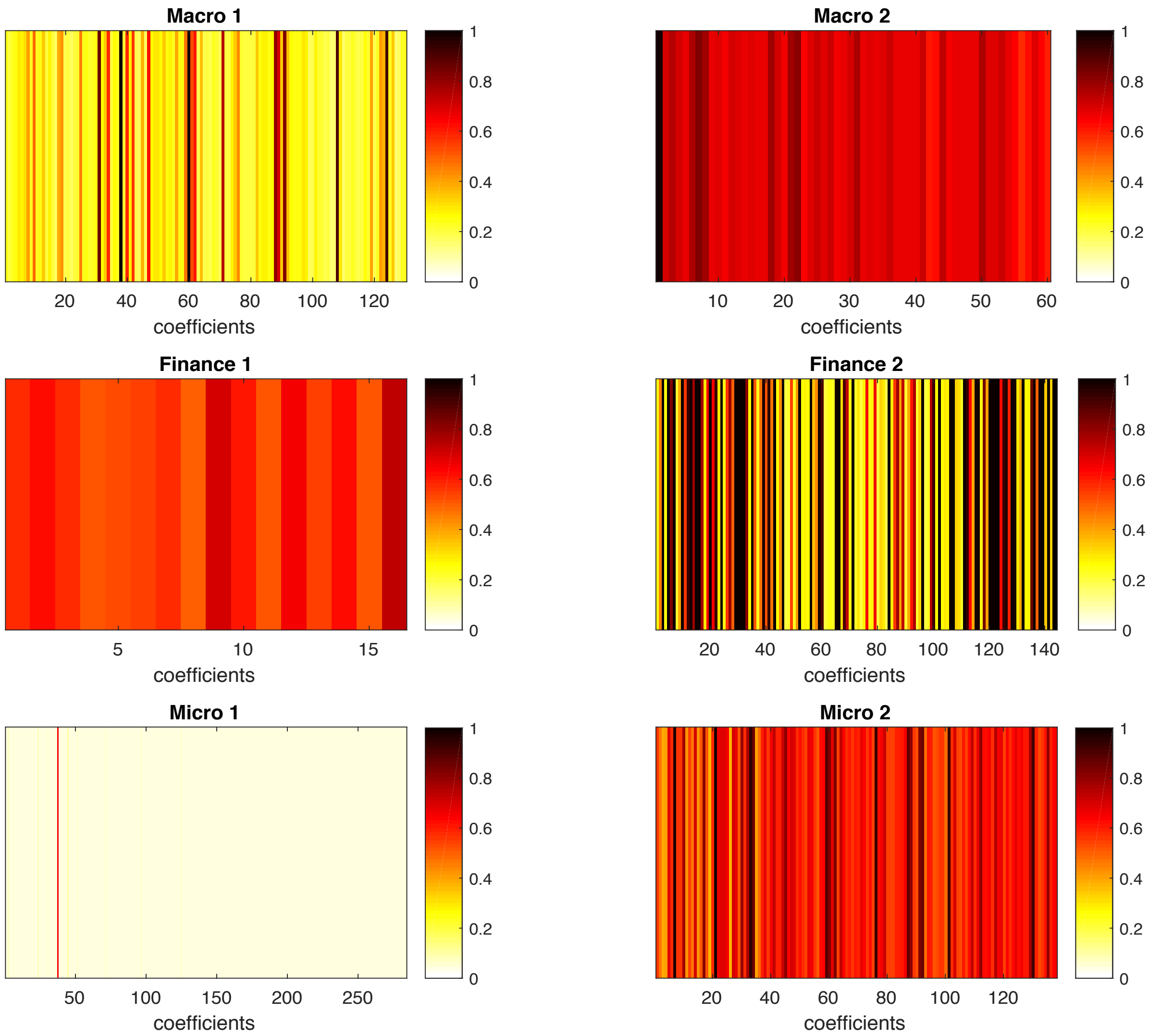
Patterns of sparsity:
Probability of inclusion of each coefficient



Best predictions with mixture of many models

- Some predictors systematically excluded?

- Predictive density implied by "model q"

$$p(y_{T+1}|Y,q) = \sum_j p\left(y_{T+1}\middle|Y,q,m_j\right) \cdot p\left(m_j\middle|Y,q\right)$$

➢ Mixture of predictive densities of many models

Exploring the posterior

0. Inclusion probability and shrinkage are complements, but imperfect

1. No clear pattern of sparsity
   - Posterior not concentrated on a single sparse model, but on a wide set
     - Predictors rarely systematically excluded
     - Model uncertainty is pervasive
     - Best predictions not with single model, but mixture of many (BMA)

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Castillo et al. (2015, Annals)

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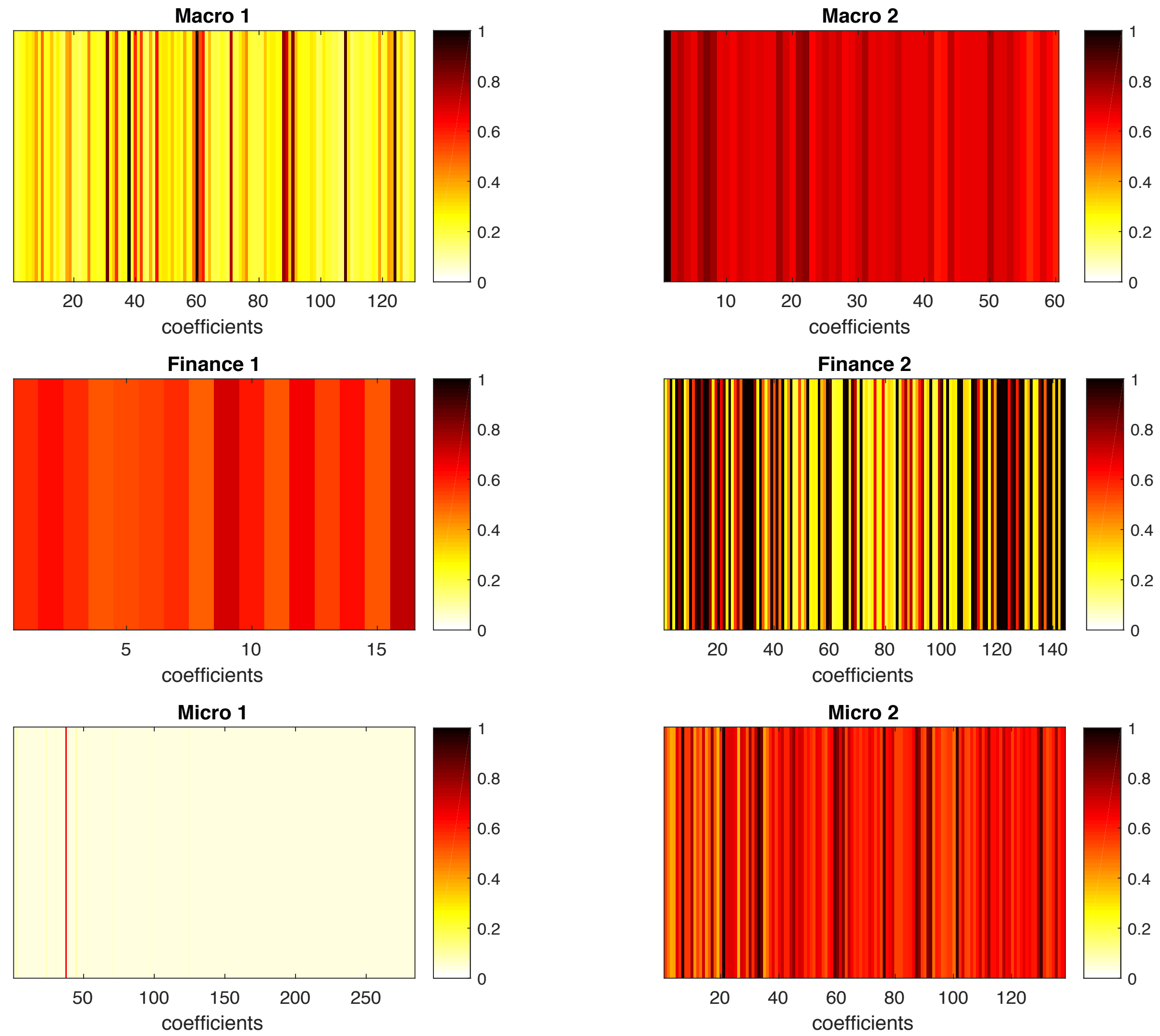
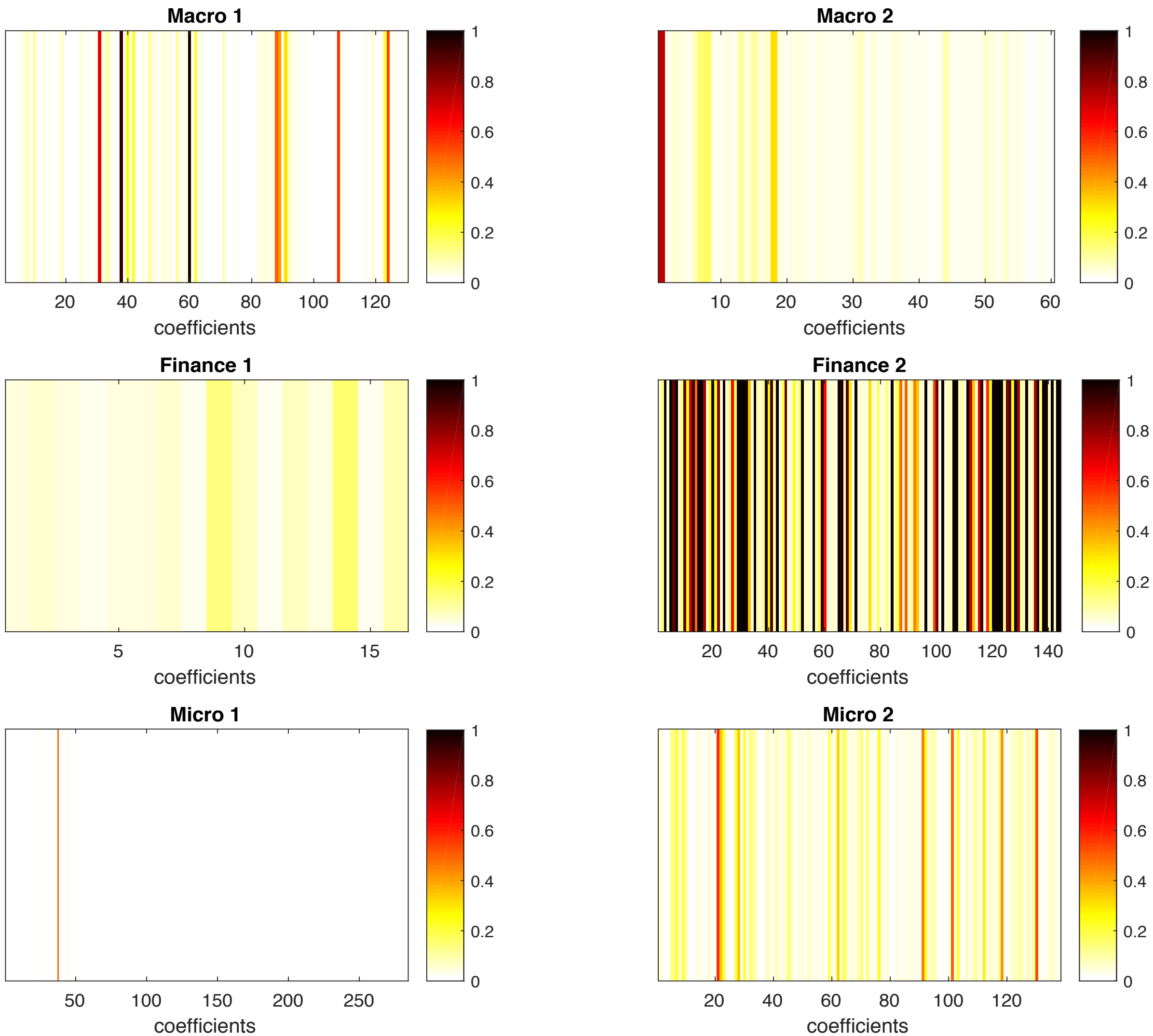
Patterns of sparsity with a flat prior on q

Probability of inclusion of each coefficient



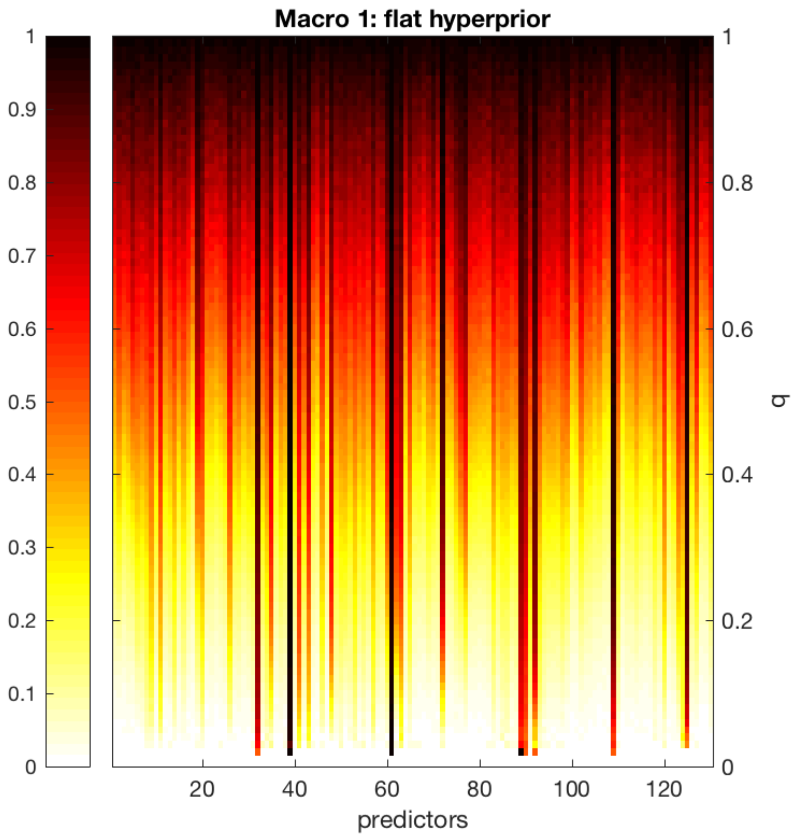
Patterns of sparsity with a tight prior on low q

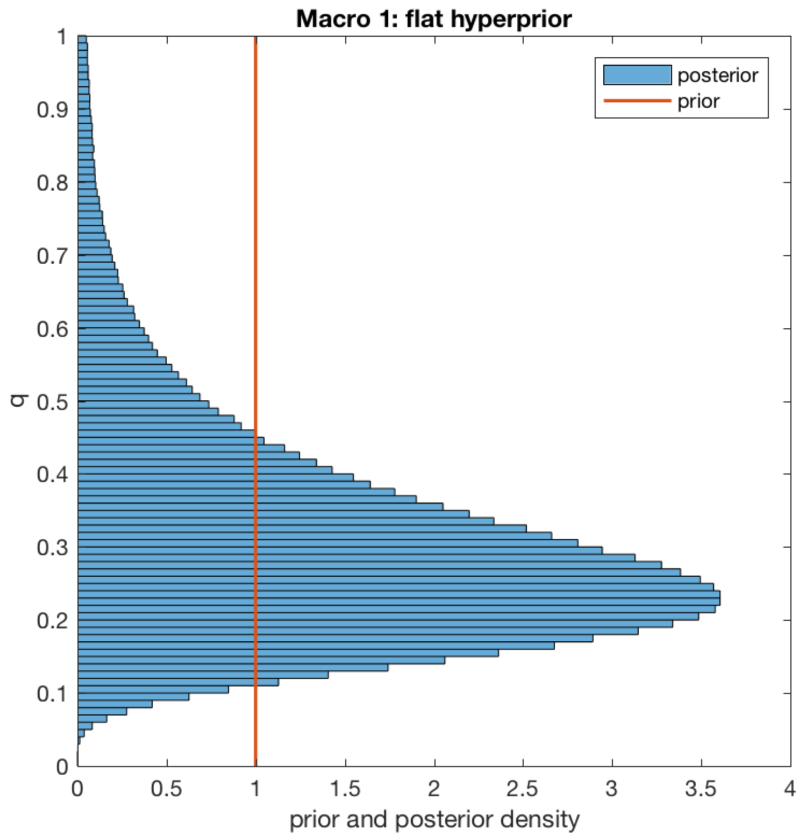
Probability of inclusion of each coefficient



Patterns of sparsity with a flat prior on q

Probability of inclusion of each coefficient, given q





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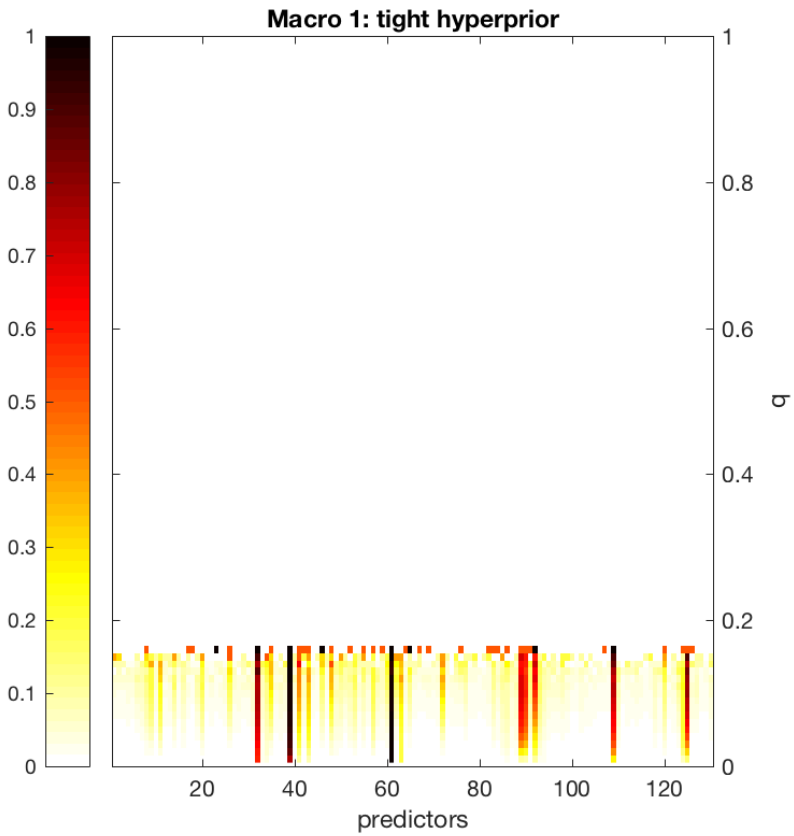
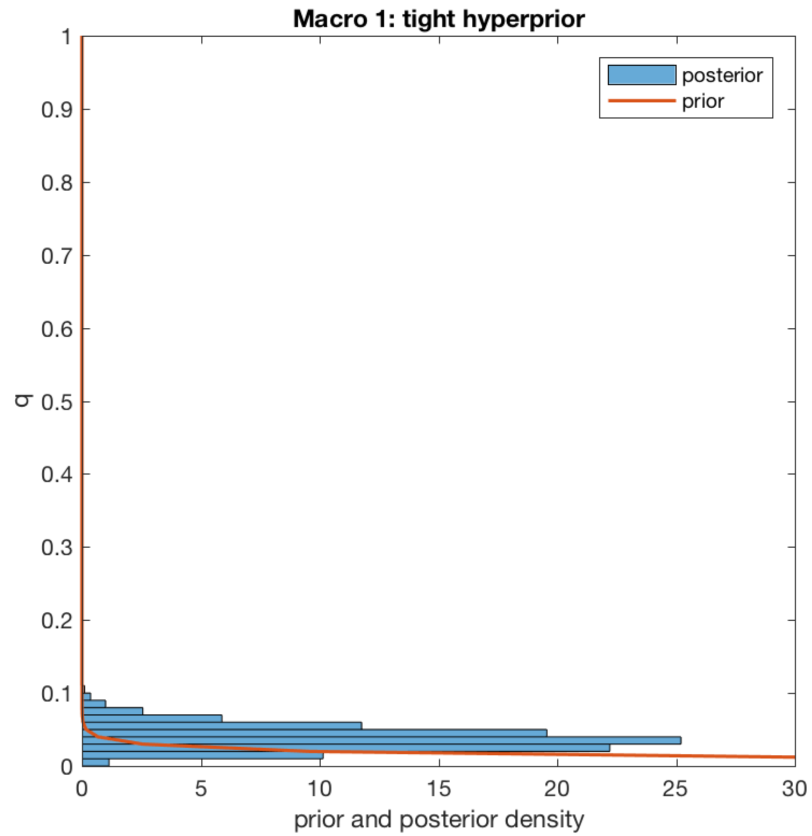
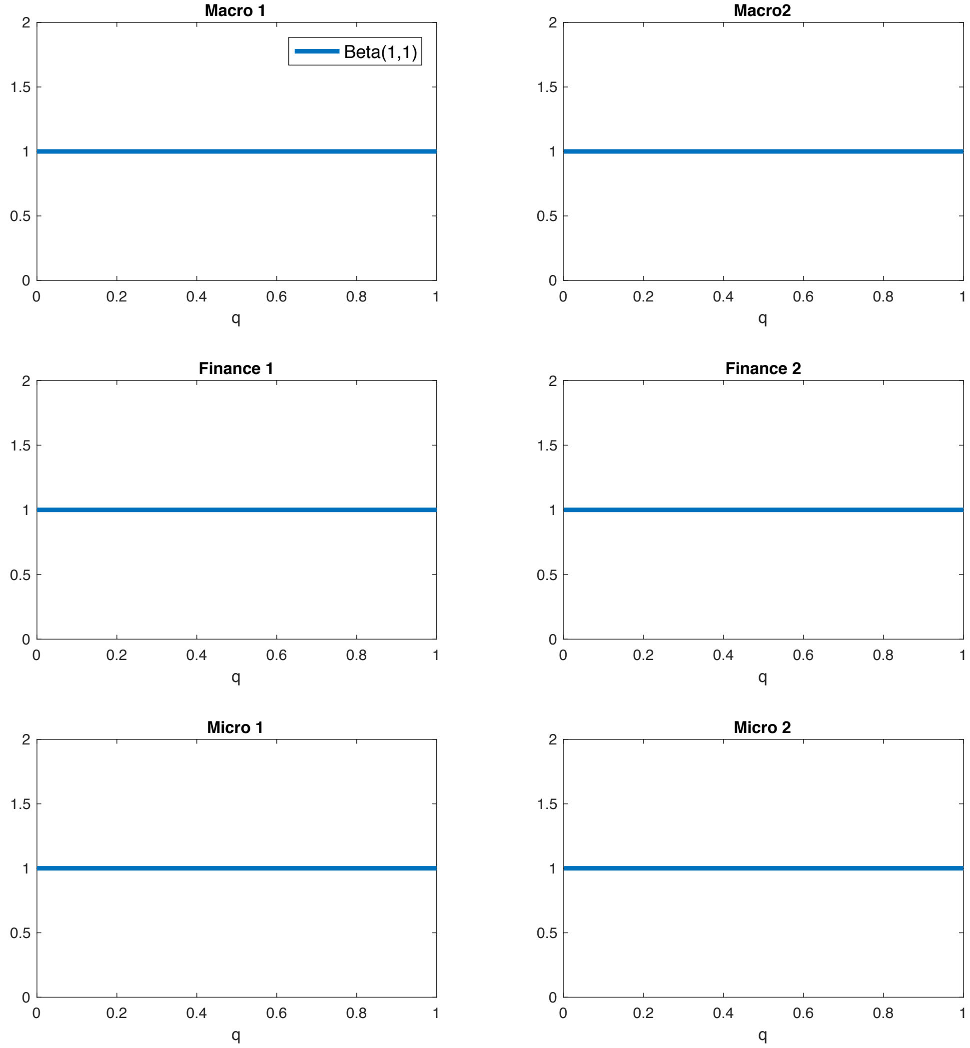
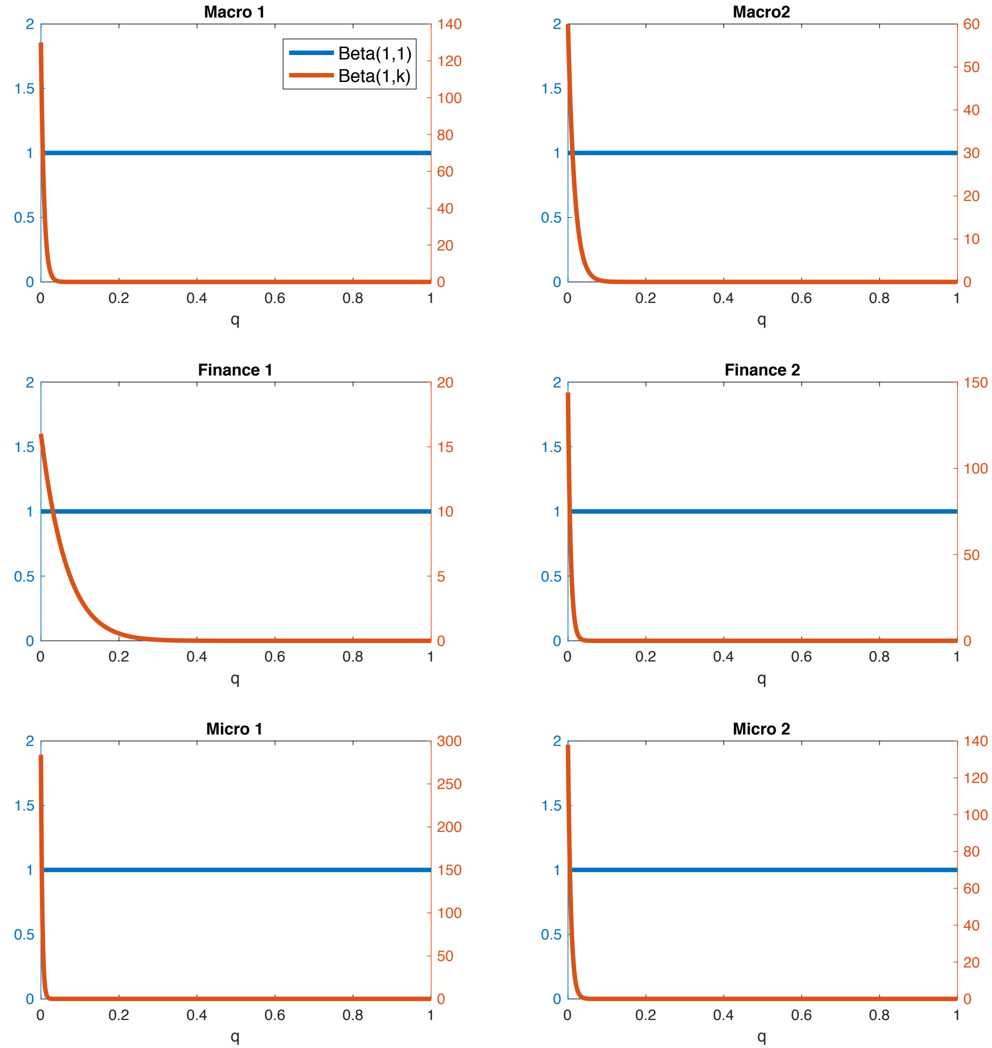




Baseline hyperprior: flat on q



Alternative hyperprior: tight on low q



Summing up

0. Inclusion probability and shrinkage are complements, but imperfect

1. No clear pattern of sparsity
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↓

The illusion of sparsity

An alternative model with 2 levels of shrinkage

$$y_t = \beta_1 x_{1t} + \cdots + \beta_k x_{kt} + \varepsilon_t \,, \;\; \varepsilon_t \underset{iid}{\sim} \mathcal{N}(0, \sigma^2)$$

- Main idea: coefficients might be **large**, **small** or **zero**

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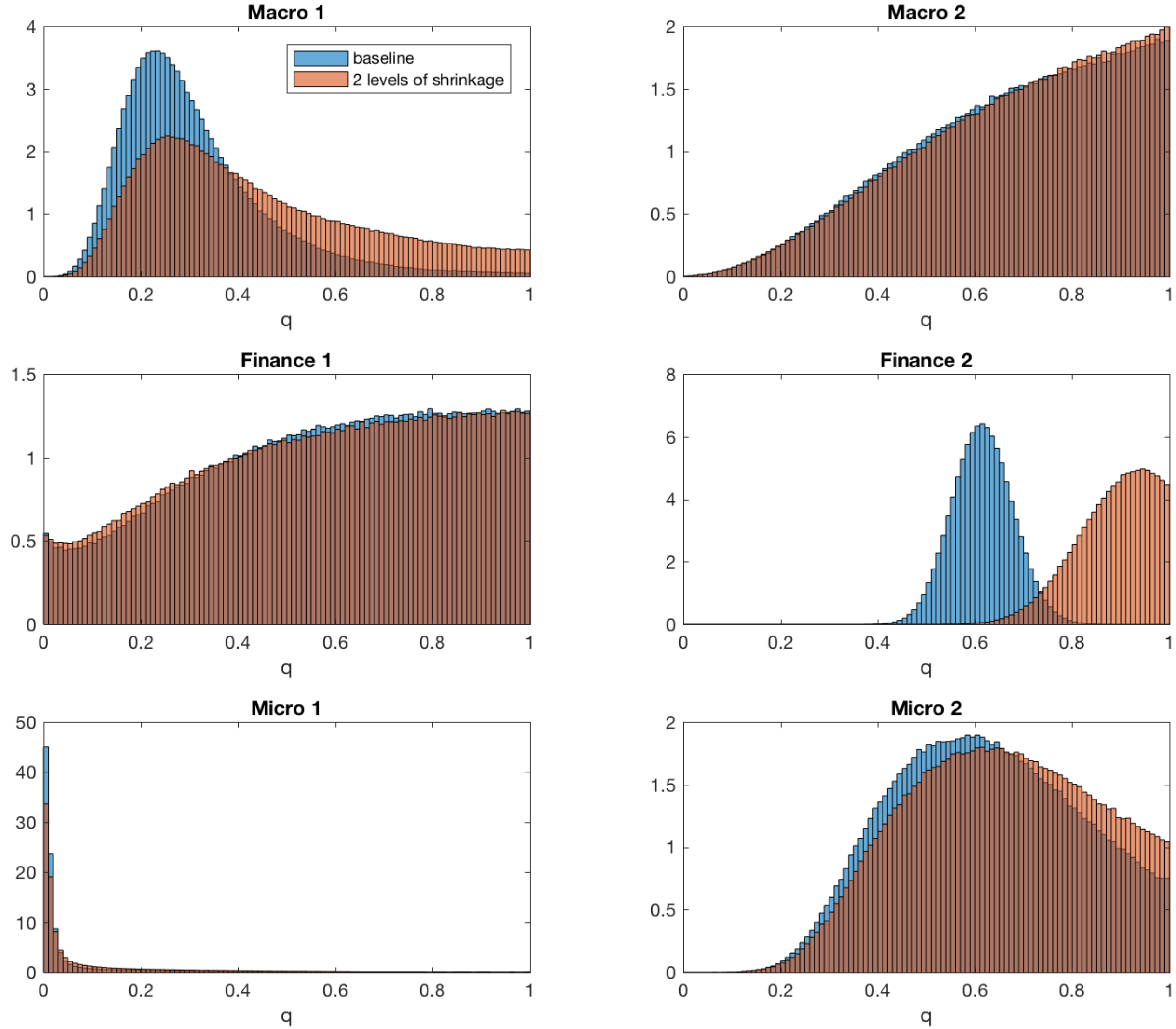
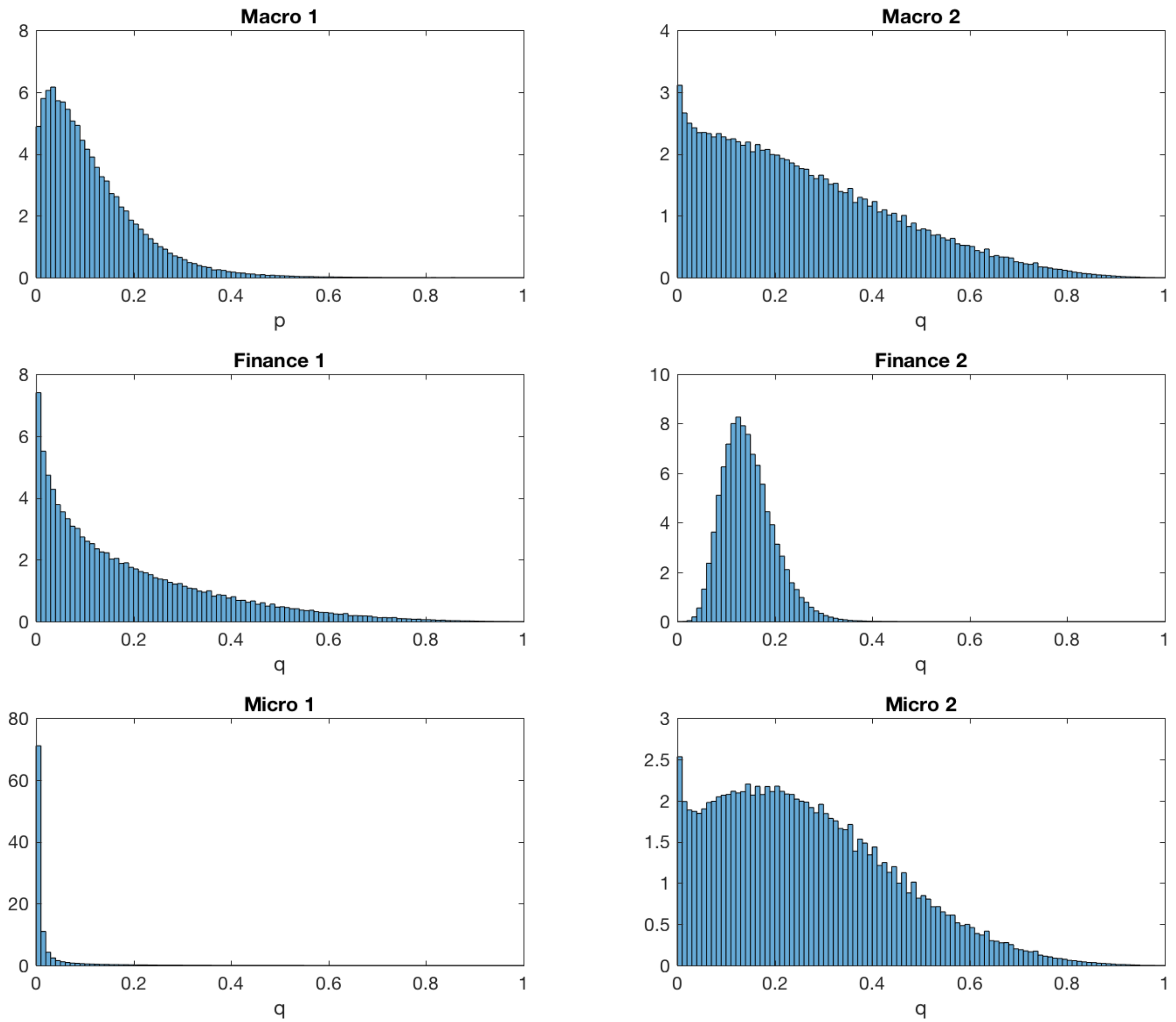
$$\beta_i | \sigma^2, \gamma^2, q \underset{iid}{\sim} \begin{cases} \begin{cases} \mathcal{N}(0, \sigma^2 \gamma_h^2) & \text{with pr.} \quad p \\ \mathcal{N}(0, \sigma^2 \gamma_l^2) & \text{with pr.} \quad 1-p \end{cases} & \text{with pr.} \quad q \\ 0 & \text{with pr.} \quad 1-q \end{cases}$$

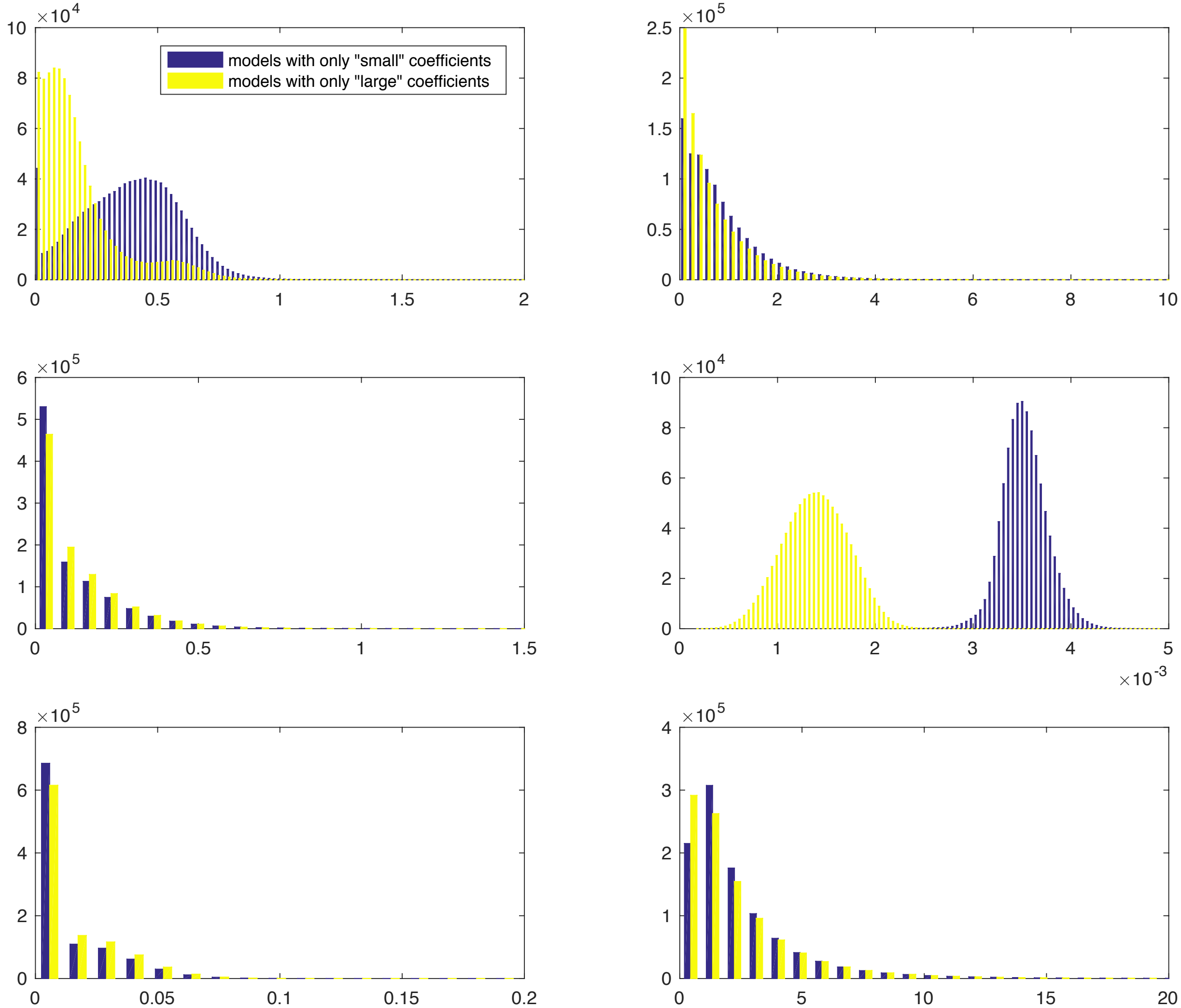
Posterior probability of inclusion, large or small



Posterior probability of inclusion, large



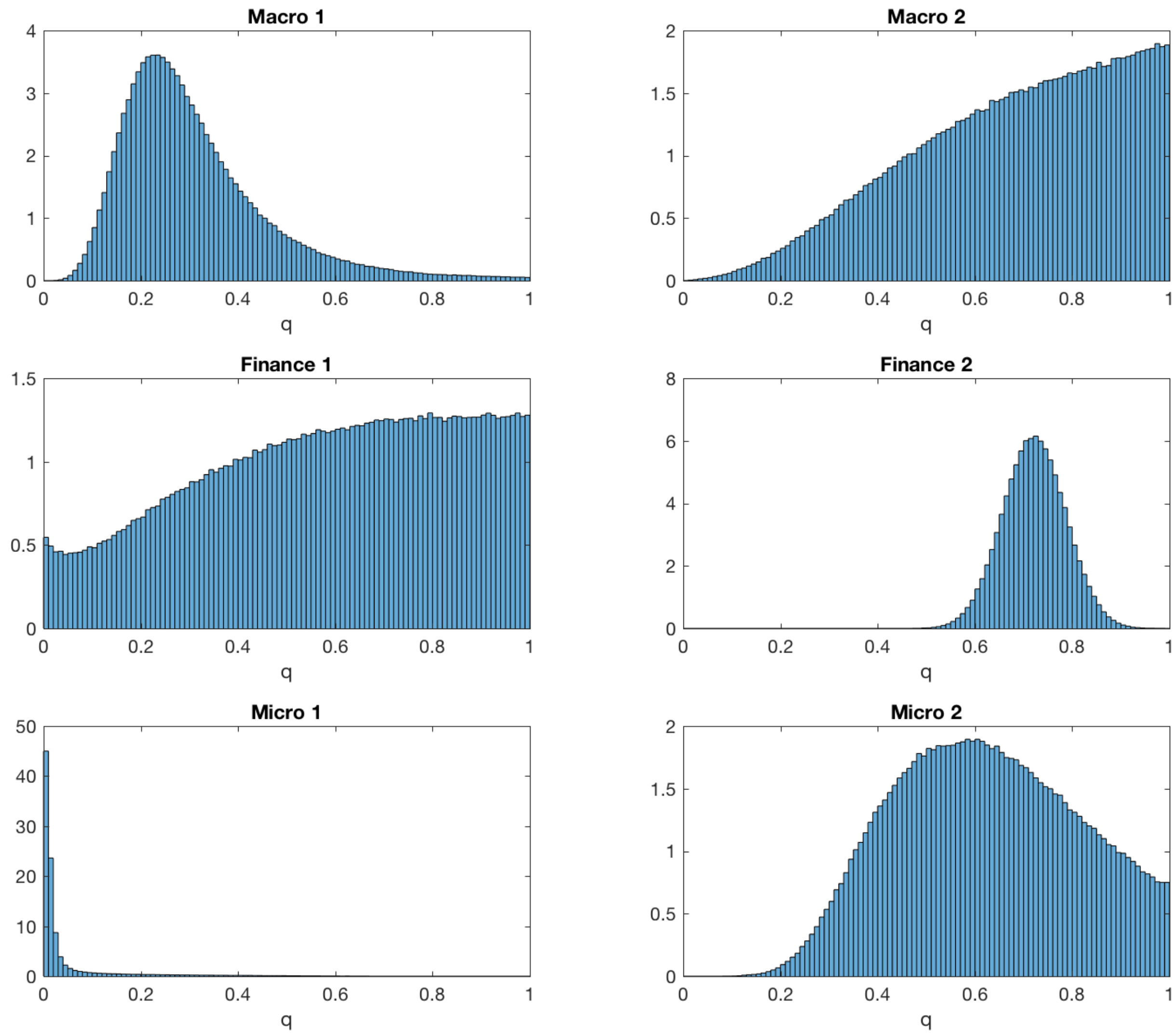
Posterior of prediction errors



“Robustness” of dense models

- Suppose want/need to pick **a single** model, not BMA
- Sometimes clear answer based on $p(q|Y)$

Posterior probability of inclusion: $p(q|Y)$



"Robustness" of dense models

- Suppose want/need to pick **a single** model, not BMA
- Sometimes clear answer based on $p(q|Y)$
- But what if $p(q|Y)$ peaks in the middle of the [0,1] interval, say 0.5?
  - high-q model ≈ low-q model

"Robustness" of dense models

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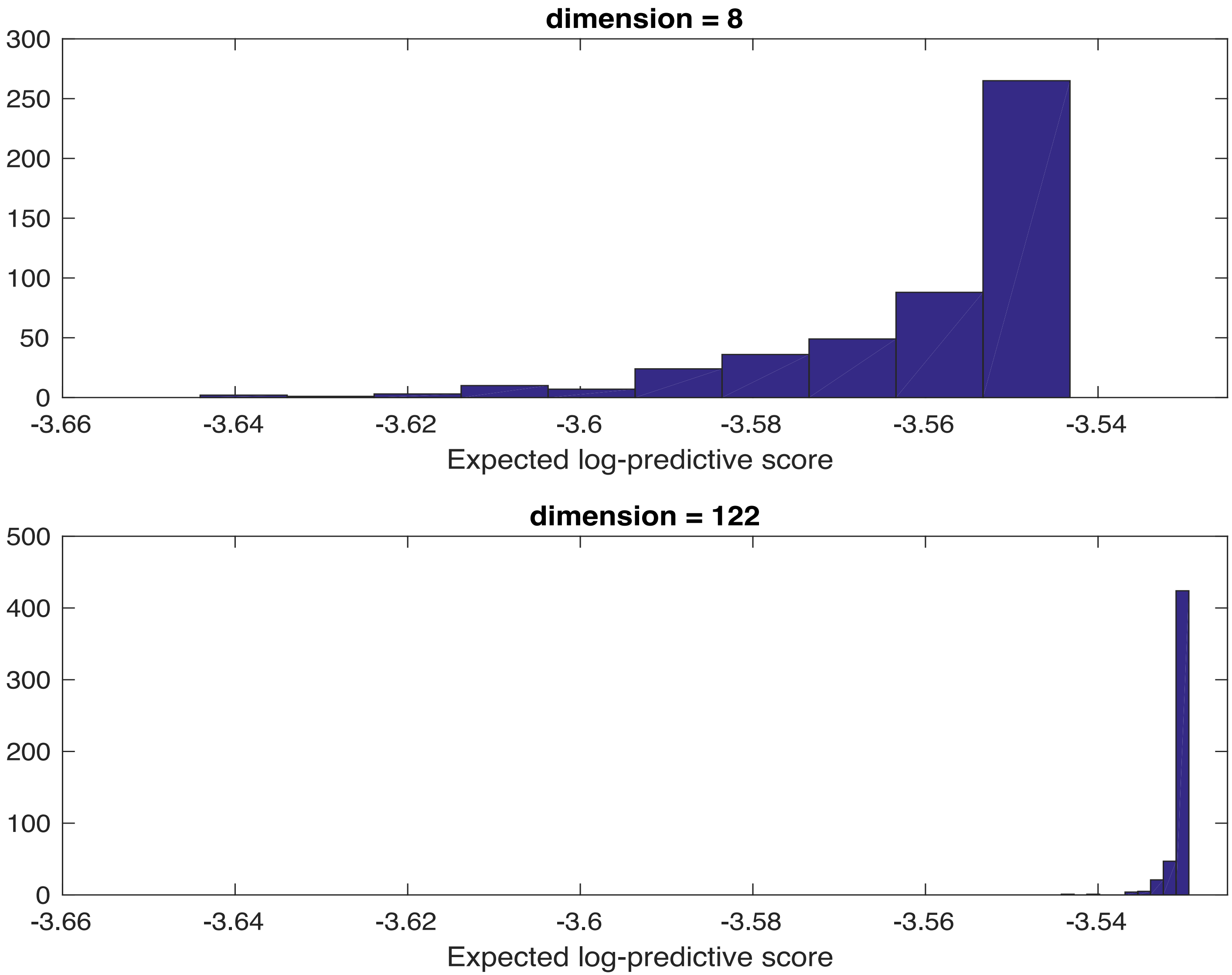
- But what if $p(q|Y)$ peaks in the middle of the [0,1] interval, say 0.5?
  - high-q model ≈ low-q model

- Intuition: Elements of high-q mixture more similar to each other and to dense model

- More formally: Expected log-predictive score

Macro 1: Expected log-predictive scores



Who’s that predictor? Macro 1

Id		Description	q	qp	q(1-p)	Lars
39		**All Employees: Nondurable goods**	**99**	68	31	2
61		**ISM : New Orders Index**	**97**	83	15	1
125		**Avg Hourly Earnings : Construction**	**92**	39	52	8
109		**ISM Manufacturing: Prices Index**	**90**	45	44	14
32		**Initial Claims**	**84**	33	54	3
92		**3-Month Treasury Bill Minus FFR**	**83**	42	40	5
89		**Moody’s Aaa Corporate Bond Yield**	**80**	39	40	12

Who’s that predictor? Finance 2

Id	Description	q	qp	LARS	FNW
31	**Short Term Reversal (L/L)**	100	100	1	7/8
144	**Std. Unexplained Volume (H/H)**	100	100	2	7/8
139	**Short Term Reversal (H/H)**	100	100	3	7/8
16	**Market Capitalization (L/L)**	100	100	4	6/8
17	**Turnover (L/L)**	100	100	16	6/8
25	**Distance from 52-week high (L/L)**	100	98	27	6/8
36	**Std. Unexplained Volume (L/L)**	100	98	5	7/8
121	**Idiosyncratic Volatility (H/H)**	100	96	26	2/8
133	**Distance from 52-week high (H/H)**	100	75	36	6/8
108	**Std. Unexplained Volume (L)**	100	75	13	7/8
4	**Book to market (L/L)**	100	48	9	1/8

Probability of inclusion (q) and R^2